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Some New a Priori Estimates for Second-Order Elliptic and Parabolic Interface Problems[1]

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We present some new a priori estimates of the solutions to the second-order elliptic and parabolic interface problems. The novelty of these estimates lies in the explicit appearance of the discontinuous coefficients and the jumps of coefficients across the interface. © 2002 Elsevier Science (USA)

Key Words: a priori estimates; interface problems; jumps in coefficients; potential theory.

1. INTRODUCTION

In this paper, we are interested in achieving some new a priori estimates of the solutions to the second-order elliptic interface problem

$$-\nabla \cdot (\beta(x)\nabla u(x)) = f(x) \qquad \text{in } \Omega \tag{1.1}$$

and the parabolic interface problem

$$\frac{\partial}{\partial t}u(x, t) - \nabla \cdot (\beta(x)\nabla u(x, t)) = f(x, t) \qquad \text{in } \Omega \times (0, T), \tag{1.2}$$

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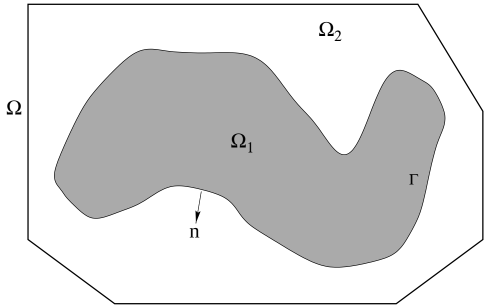

FIG. 1. Domain Ω, its subregions Ω_1, Ω_2 and interface Γ.

where Ω is a convex polyhedral domain in R^3, and $\beta(x)$ is positive and piecewise constant in Ω:

$$\beta(x) = \beta_1, \qquad x \in \Omega_1, \quad \beta(x) = \beta_2, \qquad x \in \Omega_2. \tag{1.3}$$

Here, Ω_1 is an open subdomain of Ω and lies strictly inside Ω, whereas $\Omega_2 = \Omega \backslash \Omega_1$, see Fig. 1 (for clarity, only a two-dimensional figure is shown there).

In practical applications, Ω_1 and Ω_2 may represent two distinct materials or fluids with different conductivities or diffusions. And on the interface $\Gamma = \partial\Omega_1$ between Ω_1 and Ω_2, the solution $u(x)$ satisfies the following jump conditions:

$$[u] = 0, \qquad [\beta \partial_{\mathbf{n}} u] = g \quad \text{on } \Gamma, \tag{1.4}$$

where $[v]$ stands for the jump of a function v across Γ and $\mathbf{n}$ the unit outward normal to the boundary of Ω_1. Throughout the paper, for any function v defined on Ω, we will frequently use v_1 and v_2 to denote its restrictions in Ω_1 and Ω_2, respectively, and for definiteness, we let $[v](x) = v_2(x) - v_1(x)$ for $x \in \Gamma$. On the boundary $\partial\Omega$, we assume the following homogeneous Dirichlet condition:

$$u(x) = 0 \qquad \text{on } \partial\Omega. \tag{1.5}$$

But the subsequent results can be naturally extended to the cases with non-homogeneous boundary conditions and Neumann boundary conditions.

The interface problem (1.1)–(1.5) is often encountered in material sciences and fluid dynamics. It is the case when two materials or fluids with different conductivities or diffusions are involved. Therefore it is of practical interest to study the behavior of the solution to the system (1.1)–(1.5), and in particular the effect of the discontinuous coefficient $\beta(x)$ on the solutions. Also, such behavior and effect may help numerical analysts design more efficient numerical methods. The regularities of the solutions to problem (1.1)–(1.5) and various a priori estimates of the solutions have been widely investigated in the literature. For example, for the elliptic interface problem (1.1), (1.4) and (1.5), it is well known (cf. [2, 3, 5, 12, 13, 14, 28, 29] that the solution u is of good regularity in the individual subdomains Ω_1 and Ω_2, namely, $u_i \in H^2(\Omega_i)$, $i = 1, 2$. But u has lower regularity in the entire domain Ω, usually one has only $u \in H_0^1(\Omega)$, even when the interface Γ is sufficiently smooth. And in this case, the solution u admits the following a priori estimate:

$$\|u_1\|_{H^2(\Omega_1)} + \|u_2\|_{H^2(\Omega_2)} \leqslant C(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}), \tag{1.6}$$

where C is a constant independent of u, f and g, but depends strongly and implicitly on the coefficients β_1, β_2 and the jumps in the coefficients across the interface.

On the other hand, there are also many existing numerical methods for solving the interface problem (1.1)–(1.5), see [17, 18] for finite difference methods and [5, 8] for finite element methods. Due to the non-explicit dependence of the a priori estimate (1.6) on the coefficients β_1, β_2 and their jumps across the interface, all the known error estimates of the existing numerical methods share a common weakness: it is unclear about how the accuracy of the numerical solutions is affected by the coefficients and the jumps in the coefficients.

To our knowledge, there seems little existing work in the literature, which provides the a priori estimates for the interface problem (1.1)–(1.5) with explicit appearance of the coefficients in the estimates. The purpose of this paper is to make some efforts in this direction. We will present some uniform a priori estimates, similar to (1.6), but with an explicit dependence on the coefficients β_1 and β_2, and the jumps of the coefficients across the interface. Such uniform a priori estimates, which are themselves interesting enough from the mathematical point of view, may provide us with more insights into physical behaviors of the solutions. On the other hand, the estimates may also make it possible to achieve error estimates which are uniform with respect to the jumps in the coefficients of the interface problems.

We end this section with some notations. Throughout the paper, we assume that the interface Γ is of C^2-smooth. It is well known that the solutions to the considered interface problem have no $H^2(\Omega_i)$-regularity in each individual subregion Ω_i $(i = 1, 2)$ if Γ is only piecewise smooth.

Sobolev spaces will be widely used here. For any $m \geqslant 0$, $H^m(\Omega)$ denotes the standard Sobolev spaces of mth order while $H^{-m}(\Omega)$ stands for the dual space of $H_0^m(\Omega)$. The norms and seminorms of $H^m(\Omega)$ are denoted by $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$, respectively. We refer to [1, 7, 19, 20] for more details about Sobolev spaces.

For ease of exposition, we will frequently use C to denote a generic constant, which depends only on the geometric property of Ω_1 and Ω_2. Furthermore, we shall often use the notation "$\lesssim \cdots$", which equals to "$\leqslant C \cdots$" for some generic constant C.

2. UNIFORM A PRIORI ESTIMATES FOR ELLIPTIC INTERFACE PROBLEMS

2.1. *Problem Transformation*

In this section, we confine ourselves to the elliptic interface problem (1.1). Note that the coefficients β_1 and β_2 may differ significantly in their magnitudes. This is physically more interesting, and is also our primary focus in the paper. Namely we will consider mainly the cases with either $\beta_1 \ll \beta_2$ or $\beta_2 \ll \beta_1$. Problem (1.1), (1.4) and (1.5) can be recast as follows:

$$-\Delta u_i = f/\beta_i \qquad \text{in } \Omega_i, \quad i = 1, 2, \tag{2.1}$$

$$[u] = 0, \qquad [\beta \partial_{\mathbf{n}} u] = g \quad \text{on } \Gamma, \tag{2.2}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{2.3}$$

Recall that u_1 and u_2 are the restrictions of u in Ω_1 and Ω_2, respectively, as noted earlier in the Introduction. We introduce two auxiliary functions $\tilde{u}_i \in H_0^1(\Omega_i)$, $i = 1, 2$, satisfying

$$-\Delta \tilde{u}_i = f/\beta_i \qquad \text{in } \Omega_i. \tag{2.4}$$

It is well known that (cf. [7, 20]) $\tilde{u}_i \in H^2(\Omega_i)$ and

$$\|\tilde{u}_i\|_{2,\Omega_i} \lesssim \|f\|_{0,\Omega_i}/\beta_i, \qquad i = 1, 2. \tag{2.5}$$

Now consider the following difference functions:

$$\begin{aligned} \bar{u}_i(x) &= u_i(x) - \tilde{u}_i(x), \qquad x \in \Omega_i, \\ g_1(x) &= \beta_1 \partial_{\mathbf{n}} \tilde{u}_1(x) - \beta_2 \partial_{\mathbf{n}} \tilde{u}_2(x), \qquad x \in \Gamma. \end{aligned}$$

By (2.5) and the trace theorem, we have

$$\|g_1\|_{1/2,\Gamma} \lesssim \beta_1 \|\tilde{u}_1\|_{2,\Omega_1} + \beta_2 \|\tilde{u}_2\|_{2,\Omega_2} \lesssim \|f\|_{0,\Omega}. \tag{2.6}$$

While it is easy to see from (2.1)–(2.5) that $\bar{u}_1$ and $\bar{u}_2$ satisfy

$$\Delta\bar{u}_i = 0 \qquad \text{in } \Omega_i, \quad i = 1, 2, \tag{2.7}$$

$$[\bar{u}] = 0, \qquad [\beta\partial_{\mathbf{n}}\bar{u}] = g + g_1 \quad \text{on } \Gamma, \tag{2.8}$$

$$\bar{u} = 0 \qquad \text{on } \partial\Omega. \tag{2.9}$$

This leads to the following estimates for the solution u of (2.1)–(2.3):

$$\|u\|_{2,\,\Omega_i} \lesssim \|\bar{u}_i\|_{2,\,\Omega_i} + \|f\|_{0,\,\Omega_i}/\beta_i, \qquad i = 1, 2. \tag{2.10}$$

Our subsequent task is to estimate the harmonic functions $\bar{u}_1$ and $\bar{u}_2$.

2.2. Integral Representation

We intend to use the theory of layer potentials for the estimation of the terms $\|\bar{u}_i\|_{2,\,\Omega_i}$ $(i = 1, 2)$ in (2.10). To this end, we first recall some existing results on the layer potentials, and refer to [4, 15, 26] for more details. Given a simply connected domain D with Lipschitz continuous boundary ∂D, for any density function q, the single layer potential on D is defined by

$$\mathscr{S}_D q(x) = \int_{\partial D} E(x-y)q(y)\, d\sigma_y, \qquad x \in R^3, \tag{2.11}$$

where $E(x)$ is the fundamental solution associated with the Laplacian:

$$E(x-y) = -\frac{1}{4\pi}\frac{1}{|x-y|},$$

and $d\sigma_y$ is the surface measure. In the subsequent analysis, we shall also frequently use $\mathscr{S}_D$, restricted on ∂D, as a boundary integral operator on ∂D. For ease of notation, we will still use the same notation $\mathscr{S}_D$ for this restriction operator when there is no confusion caused. For a function v defined in R^3, we denote

$$v^{\pm}(x) = \lim_{t\to 0^+} v(x \pm t\mathbf{n}_x), \qquad x \in \partial D,$$

$$\partial_{\mathbf{n}^\pm} v(x) = \lim_{t\to 0^+} \langle \mathbf{n}_x, \nabla v(x \pm t\mathbf{n}_x)\rangle, \qquad x \in \partial D$$

when the related limits exist. Here $\mathbf{n}_x$ is the unit outward normal to ∂D at the point x. We have the classical trace formulas (cf. [4, 15, 21, 22, 26]):

$$(\mathscr{S}_D q)^{\pm}(x) = \mathscr{S}_D q(x), \tag{2.12}$$

$$\partial_{\mathbf{n}^\pm}\mathscr{S}_D q(x) = (\pm\tfrac{1}{2} I + \mathscr{K}_D^*)q(x), \tag{2.13}$$

where $\mathscr{K}_D$ is the integral operator given by

$$\mathscr{K}_D q(x) = \frac{1}{4\pi}\,\text{p.v.}\int_{\partial D} \frac{\langle \mathbf{n}_y, y - x\rangle}{|x-y|^3}\, q(y)\, d\sigma_y,$$

and $\mathscr{K}_D^*$ is the L^2-adjoint of $\mathscr{K}_D$, that is,

$$\mathscr{K}_D^* q(x) = \frac{1}{4\pi}\,\text{p.v.}\int_{\partial D} \frac{\langle \mathbf{n}_x, x - y\rangle}{|x-y|^3}\, q(y)\, d\sigma_y.$$

For the later use, we now list some known results about these integral operators. For the basic notation and theory about pseudo-differential operators used below, we refer to [23, 24, 25].

LEMMA 2.1. *For the operators $\mathscr{S}_D$, $\mathscr{K}_D$ and $\mathscr{K}_D^*$, we have*

1. *For any real number λ with $|\lambda| > \frac{1}{2}$, $\lambda I - \mathscr{K}_D^*$ is an invertible operator on $L^2(\partial D)$* (*cf.* [6]).

2. *$\mathscr{S}_D$ maps $L^2(\partial D)$ into $H^1(\partial D)$, and has a bounded inverse* (*cf.* [15, *p.* 56]).

3. *$\mathscr{S}_D$, $\mathscr{K}_D$ and $\mathscr{K}_D^*$ are all the pseudo-differential operators of order -1 on the compact manifold ∂D, the principal symbol of $\mathscr{S}_D$ is $-\frac{1}{2}|\xi|^{-1}$* (*cf.* [4, *pp.* 87–93]).

With the above preparations, we can now characterize the solution $\bar{u}$ to system (2.7)–(2.9) by some simple layer potentials.

LEMMA 2.2. *The solution $\bar{u}$ of problem* (2.7)–(2.9) *can be characterized by*

$$\bar{u}(x) = (\mathscr{S}_{\Omega_1}\phi)(x) - (\mathscr{S}_{\Omega}\psi)(x), \qquad x \in \Omega, \tag{2.14}$$

where $\psi = \partial_\nu \bar{u}_2$ on $\partial\Omega$ with ν being the unit outward normal to $\partial\Omega$, and $\phi \in H^{1/2}(\Gamma)$ solves

$$\left(\frac{\beta_1+\beta_2}{2(\beta_1-\beta_2)}\, I - \mathscr{K}_{\Omega_1}^*\right)\phi = -\partial_{\mathbf{n}}(\mathscr{S}_\Omega\psi) + \frac{1}{\beta_1-\beta_2}\,(g+g_1) \qquad \textit{on } \Gamma. \tag{2.15}$$

Proof. We first show that the difference on the right-hand side of (2.14) makes sense. For this, it suffices to verify that the integral equation (2.15) has a unique solution $\phi \in L^2(\Gamma)$. By the definitions of $\mathscr{S}_\Omega\psi$ and g_1, we know that the right-hand side of (2.15) lies in $H^{1/2}(\Gamma)$. Then the desired result follows immediately from the first statement in Lemma 2.1. Moreover, noting that $\frac{\beta_1+\beta_2}{2(\beta_1-\beta_2)}\, I - \mathscr{K}_{\Omega_1}^*$ is a pseudo-differential operator of order 0 with the principal symbol $\frac{\beta_1+\beta_2}{2(\beta_1-\beta_2)}$, it is then an elliptic operator of order 0. By the basic theory of pseudo-differential operators, we further have $\phi \in H^{1/2}(\Gamma)$.

Secondly, applying a similar technique to the one as used in [10, 11], one can show that the following problem has at most one solution in $H^1(R^3)$:

$$\Delta w = 0 \qquad \text{in } \Omega_1 \cup \Omega_2 \cup (R^3 \backslash \bar{\Omega}), \tag{2.16}$$

$$[w] = 0, \qquad [\beta \partial_{\mathbf{n}} w] = g + g_1 \qquad \text{on } \Gamma, \tag{2.17}$$

$$w_2^- - w_2^+ = 0, \qquad \frac{\partial w_2^-}{\partial v} - \frac{\partial w_2^+}{\partial v} = \psi \quad \text{on } \partial\Omega. \tag{2.18}$$

By the evaluation formulas (2.12) and (2.13) we know that

$$R_1(x) = \mathscr{S}_{\Omega_1}\phi(x) - \mathscr{S}_{\Omega}\psi(x)$$

is a solution of (2.16)–(2.18). On the other hand, it is easy to verify that

$$R_2(x) = \begin{cases} \bar{u}(x) & \text{in } \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

also satisfies (2.16)–(2.18). This implies equality (2.14). ∎

Remark 2.1. The potential representation (2.14) for the piecewise harmonic function $\bar{u}$ was initiated by [10, 11], where the representation plays a crucial role in a different context, the identification of discontinuous conductivity coefficients. The derivation of such a representation in the Lipschitz boundary case is very technical (cf. [10, 11]). For the simpler case of C^2-smooth boundary as needed here, one can find a different and more intuitive derivation of the representation, see [9].

2.3. *A Priori Estimates*

We are now ready to derive the a priori estimates on the solution u to system (2.1)–(2.3). For this, it suffices to estimate $\bar{u}_1$ and $\bar{u}_2$ of the solutions to (2.7)–(2.9) by using (2.10). Since $\bar{u}_1$ and $\bar{u}_2$ are harmonic in Ω_1 and Ω_2, respectively, we have by (2.15) that

$$\|\bar{u}_1\|_{2,\Omega_1} + \|\bar{u}_2\|_{2,\Omega_2} \lesssim \|\bar{u}\|_{3/2,\Gamma} \lesssim \|\mathscr{S}_{\Omega_1}\phi\|_{3/2,\Gamma} + \|\mathscr{S}_{\Omega}\psi\|_{3/2,\Gamma}. \tag{2.19}$$

Since $\mathscr{S}_{\Omega_1}$ is an invertible pseudo-differential operator of order -1, we have (cf. [4, p. 262])

$$\|\mathscr{S}_{\Omega_1}\phi\|_{3/2,\Gamma} \lesssim \|\phi\|_{1/2,\Gamma}. \tag{2.20}$$

On the other hand, for any C^2-smooth surface $\Gamma' \subset\subset \Omega$ and $x \in \Gamma'$, the kernel function $E(x-y)$ of the operator $\mathscr{S}_{\Omega}$ is C^∞-smooth. So we can directly see

$$\|\mathscr{S}_\Omega\psi\|_{3/2,\Gamma'} \lesssim \|\mathscr{S}_\Omega\psi\|_{2,\Gamma'} \lesssim \|\psi\|_{\alpha,\partial\Omega} \qquad \forall\alpha \in R. \tag{2.21}$$

This together with (2.19) and (2.20) implies

$$\|\bar{u}_1\|_{2,\,\Omega_1} + \|\bar{u}_2\|_{2,\,\Omega_2} \lesssim \|\phi\|_{1/2,\Gamma} + \|\psi\|_{0,\partial\Omega}. \tag{2.22}$$

Further, it follows from the condition $\bar{u}_2 = 0$ on $\partial\Omega$ and (2.14) that

$$(\mathscr{S}_{\Omega_1}\phi)(x) = (\mathscr{S}_\Omega\psi)(x), \qquad x \in \partial\Omega,$$

then by the isomorphism of S_Ω (cf. [4]) and the same argument as for (2.21) we obtain

$$\|\psi\|_{0,\,\partial\Omega} \lesssim \|\mathscr{S}_\Omega\psi\|_{1,\,\partial\Omega} = \|\mathscr{S}_{\Omega_1}\phi\|_{1,\,\partial\Omega} \lesssim \|\phi\|_{\alpha,\,\Gamma} \qquad \forall\alpha \in R.$$

This with (2.22) yields

$$\|\bar{u}_1\|_{2,\,\Omega_1} + \|\bar{u}_2\|_{2,\,\Omega_2} \lesssim \|\phi\|_{1/2,\,\Gamma}. \tag{2.23}$$

Now it remains to bound $\|\phi\|_{1/2,\Gamma}$. It follows from (2.15) that

$$\begin{aligned}\frac{\beta_1+\beta_2}{2|\beta_1-\beta_2|}\|\phi\|_{1/2,\,\Gamma} \leqslant \|\mathscr{K}^*_{\Omega_1}\phi\|_{1/2,\,\Gamma} + \|\partial_{\mathbf{n}}(\mathscr{S}_\Omega\psi)\|_{1/2,\,\Gamma}& \\ + \frac{1}{|\beta_1-\beta_2|}(\|g\|_{1/2,\,\Gamma} + \|g_1\|_{1/2,\,\Gamma}).&\end{aligned} \tag{2.24}$$

By Lemma 2.1 we know $\mathscr{K}^*_{\Omega_1}$ is a pseudo-differential operator of order -1, which implies (cf. [4, 23])

$$\|\mathscr{K}^*_{\Omega_1}\phi\|_{1/2,\,\Gamma} \lesssim \|\phi\|_{-1/2,\,\Gamma}. \tag{2.25}$$

We next choose a domain Ω' such that $\Omega' \subset\subset \Omega$, with Γ as its interior boundary and $\partial\Omega' \in C^2$ as its exterior boundary, respectively. Since $\mathscr{S}_\Omega\psi$ is harmonic in Ω, we obtain from (2.21) and the basic regularity estimates for elliptic operators (cf. [7, 20]) that

$$\begin{aligned}\|\partial_{\mathbf{n}}(\mathscr{S}_\Omega\psi)\|_{1/2,\,\Gamma} &\lesssim \|\mathscr{S}_\Omega\psi\|_{2,\,\Omega'} \lesssim \|\mathscr{S}_\Omega\psi\|_{3/2,\,\Gamma} + \|\mathscr{S}_\Omega\psi\|_{3/2,\,\partial\Omega'} \\ &\lesssim \|\psi\|_{-1/2,\,\partial\Omega} = \|\partial_\nu\bar{u}_2\|_{-1/2,\,\partial\Omega}.\end{aligned} \tag{2.26}$$

Now for any $\eta \in H^{1/2}(\partial\Omega)$, we introduce an auxiliary function $v_\eta \in H^1(\Omega_2)$ satisfying

$$\begin{aligned}\Delta v_\eta &= 0 \qquad \text{in } \Omega_2,\\ v_\eta &= 0 \qquad \text{on } \Gamma,\\ v_\eta &= \eta \qquad \text{on } \partial\Omega.\end{aligned}$$

It is clear that $\|v_\eta\|_{1,\,\Omega_2} \lesssim \|\eta\|_{1/2,\,\partial\Omega}$ and

$$(\partial_\nu \bar{u}_2, v_\eta)_{\partial\Omega} = \int_{\Omega_2} \nabla \bar{u}_2 \cdot \nabla v_\eta \, dx,$$

where $(\cdot,\cdot)_{\partial\Omega}$ denotes the dual product between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. We then have

$$|(\partial_\nu \bar{u}_2, \eta)_{\partial\Omega}| = |(\partial_\nu \bar{u}_2, v_\eta)_{\partial\Omega}| \leqslant \|\nabla \bar{u}_2\|_{0,\,\Omega_2} \|\nabla v_\eta\|_{0,\,\Omega_2} \lesssim \|\nabla \bar{u}_2\|_{1,\,\Omega_2} \|\eta\|_{1/2,\,\partial\Omega},$$

which directly leads to

$$\|\partial_\nu \bar{u}_2\|_{-1/2,\,\partial\Omega} \lesssim \|\nabla \bar{u}_2\|_{1,\,\Omega_2} \lesssim \|\bar{u}\|_{1/2,\,\Gamma}. \tag{2.27}$$

Combining (2.24)–(2.27) and (2.6) we obtain

$$\frac{\beta_1 + \beta_2}{2|\beta_1 - \beta_2|} \|\phi\|_{1/2,\,\Gamma} \lesssim \|\phi\|_{-1/2,\,\Gamma} + \|\bar{u}\|_{1/2,\,\Gamma} + \frac{1}{|\beta_1 - \beta_2|} (\|g\|_{1/2,\,\Gamma} + \|f\|_{0,\,\Omega}). \tag{2.28}$$

Moreover, by the representation (2.14) we know

$$\mathscr{S}_{\Omega_1} \phi = \bar{u} + \mathscr{S}_\Omega \psi \qquad \text{on } \Gamma,$$

which together with (2.21) and (2.27) and the fact that $\mathscr{S}_{\Omega_1}$ is an isomorphism from $H^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$ (cf. [4, p. 258]) yields

$$\begin{aligned} \|\phi\|_{-1/2,\,\Gamma} &\lesssim \|\mathscr{S}_{\Omega_1}\phi\|_{1/2,\,\Gamma} \lesssim \|\bar{u}\|_{1/2,\,\Gamma} + \|\mathscr{S}_\Omega \psi\|_{1/2,\,\Gamma} \\ &\lesssim \|\bar{u}\|_{1/2,\,\Gamma} + \|\psi\|_{-1/2,\,\Gamma} \lesssim \|\bar{u}\|_{1/2,\,\Gamma}. \end{aligned} \tag{2.29}$$

Combining this with (2.28) gives

$$\frac{\beta_1 + \beta_2}{2|\beta_1 - \beta_2|} \|\phi\|_{1/2,\,\Gamma} \lesssim \|\bar{u}\|_{1/2,\,\Gamma} + \frac{1}{|\beta_1 - \beta_2|} (\|g\|_{1/2,\,\Gamma} + \|f\|_{0,\,\Omega}). \tag{2.30}$$

We now proceed to estimate $\|\bar{u}\|_{1/2,\,\Gamma}$. It is easy to see from (2.7)–(2.9) that $\bar{u} \in H_0^1(\Omega)$ and solves

$$\int_{\Omega_1} \beta_1 \nabla \bar{u}_1 \cdot \nabla v_1 \, dx + \int_{\Omega_2} \beta_2 \nabla \bar{u}_2 \cdot \nabla v_2 \, dx = -\int_\Gamma (g + g_1) v \, ds \qquad \forall v \in H_0^1(\Omega).$$

Taking $v = \bar{u}$, and noting $\bar{u}_i$ are harmonic in Ω_i, $i = 1, 2$, we have by the extension theorem (cf. [27]) that

$$\begin{aligned} \beta_2 \|\bar{u}\|^2_{1/2,\,\Gamma} &\lesssim \int_{\Omega_1} \beta_1 \nabla \bar{u}_1 \cdot \nabla \bar{u}_1 dx + \int_{\Omega_2} \beta_2 \nabla \bar{u}_2 \cdot \nabla \bar{u}_2 \, dx \\ &\lesssim \|\bar{u}\|_{1/2,\,\Gamma} (\|g\|_{-1/2,\,\Gamma} + \|g_1\|_{-1/2,\,\Gamma}), \end{aligned}$$

that is,

$$\|\bar{u}\|_{1/2,\Gamma} \lesssim \frac{1}{\beta_2}\left(\|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega}\right), \tag{2.31}$$

where we have used the basic fact that $\|g_1\|_{-1/2,\Gamma} \lesssim \|f\|_{-1,\Omega}$, which can be easily verified from the definition of g_1 and the same duality argument as used for deriving (2.27).

Plugging (2.31) into (2.30) leads to

$$\frac{\beta_1+\beta_2}{2|\beta_1-\beta_2|}\|\phi\|_{1/2,\Gamma} \lesssim \frac{1}{\beta_2}\left(\|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega}\right) + \frac{1}{|\beta_1-\beta_2|}\left(\|g\|_{1/2,\Gamma} + \|f\|_{0,\Omega}\right),$$

or equivalently,

$$\begin{aligned}\|\phi\|_{1/2,\Gamma} \lesssim & \frac{2|\beta_1-\beta_2|}{\beta_1+\beta_2}\frac{1}{\beta_2}\left(\|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega}\right) \\ & + \frac{2}{\beta_1+\beta_2}\left(\|g\|_{1/2,\Gamma} + \|f\|_{0,\Omega}\right).\end{aligned} \tag{2.32}$$

Now letting k be the jump of the coefficient across the interface, namely, $k = \beta_1/\beta_2$, then by (2.10), (2.23) and (2.32) we have derived the following theorem:

THEOREM 2.1. *In the case that the coefficient function* $\beta(x)$ *of the interface problem* (1.1), (1.4) *and* (1.5) *is piecewise constant*, *see* (1.3), *the solution* u *satisfies the following a priori estimates*:

$$\begin{aligned}\beta_1\|u_1\|_{2,\Omega_1} \lesssim & \|f\|_{0,\Omega_1} + \frac{2|k-1|k}{k+1}\left(\|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega}\right) \\ & + \frac{2k}{1+k}\left(\|g\|_{1/2,\Gamma} + \|f\|_{0,\Omega}\right),\end{aligned} \tag{2.33}$$

$$\begin{aligned}\beta_2\|u_2\|_{2,\Omega_2} \lesssim & \|f\|_{0,\Omega_2} + \frac{2|k-1|}{k+1}\left(\|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega}\right) \\ & + \frac{2}{1+k}\left(\|g\|_{1/2,\Gamma} + \|f\|_{0,\Omega}\right).\end{aligned} \tag{2.34}$$

COROLLARY 2.1. *Under the assumption of Theorem* 2.1, *the solution* u *of the interface problem* (1.1), (1.4) *and* (1.5) *satisfies the a priori estimates*

$$\beta_1\|u_1\|_{2,\Omega_1} \lesssim \|g\|_{1/2,\Gamma} + \|f\|_{0,\Omega}, \qquad \beta_2\|u_2\|_{2,\Omega_2} \lesssim \|g\|_{1/2,\Gamma} + \|f\|_{0,\Omega}, \tag{2.35}$$

if $\beta_1 < \beta_2$, *and*

$$\beta_1\|u_1\|_{2,\Omega_1} \lesssim k\left(\|g\|_{1/2,\Gamma} + \|f\|_{0,\Omega}\right), \quad \beta_2\|u_2\|_{2,\Omega_2} \lesssim \|g\|_{1/2,\Gamma} + \|f\|_{0,\Omega}. \tag{2.36}$$

if $\beta_2 < \beta_1$.

Remark. Physically one is more interested in the cases that the jumps in the coefficients across the interface are very large, that is, either $\beta_1 \ll \beta_2$ or $\beta_2 \ll \beta_1$. Corollary 2.1 provides some important information about the effects of the coefficients on the behaviors of the solutions to the interface problem (1.1), (1.4) and (1.5). In the sense of H^2-norm, the (conductivity or diffusion) coefficient β_1 has influence mainly on the behavior of the solution in the region Ω_1, and has no direct influence on the behavior of the solution in the region Ω_2, no matter whether $\beta_1 \ll \beta_2$ or $\beta_1 \gg \beta_2$. On the other hand, β_2 has influence only on the behavior of the solution in the region Ω_2 when $\beta_1 \ll \beta_2$. But it has direct influence on the behavior of the solution both in the regions Ω_1 and Ω_2 when $\beta_1 \gg \beta_2$.

In the energy-norm case, we have also similar results as stated in the following theorem:

THEOREM 2.2. *Under the assumption of Theorem* 2.1, *the solution u of the interface problem* (1.1), (1.4) *and* (1.5) *satisfies the a priori estimates*

$$\begin{aligned} \beta_1 \|u_1\|_{1,\Omega_1} &\lesssim \|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega}, \\ \beta_2 \|u_2\|_{1,\Omega_2} &\lesssim \|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega}, \end{aligned} \tag{2.37}$$

if $\beta_1 < \beta_2$, *and*

$$\begin{aligned} \beta_1 \|u_1\|_{1,\Omega_1} &\lesssim k(\|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega}), \\ \beta_2 \|u_2\|_{1,\Omega_2} &\lesssim \|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega}, \end{aligned} \tag{2.38}$$

if $\beta_2 < \beta_1$.

Proof. Noting the definition of $\tilde{u}_i$ (cf. (2.4)), we easily see

$$\|\tilde{u}\|_{1,\Omega_i} \lesssim \frac{1}{\beta_i} \|f\|_{-1,\Omega_i}, \qquad i = 1, 2. \tag{2.39}$$

Since $\bar{u}_i$ is harmonic in Ω_i, by (2.31) and (2.39) we have

$$\begin{aligned} \|u_2\|_{1,\Omega_2} &\lesssim \|\tilde{u}_2\|_{1,\Omega_2} + \|\bar{u}_2\|_{1,\Omega_2} \lesssim \|\tilde{u}_2\|_{1,\Omega_2} + \|\bar{u}\|_{1/2,\Gamma} \\ &\lesssim \frac{1}{\beta_2} \|f\|_{-1,\Omega_2} + \frac{1}{\beta_2} (\|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega}), \end{aligned}$$

which gives

$$\beta_2 \|u_2\|_{1,\Omega_2} \lesssim \|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega}.$$

Similarly, we obtain

$$\|u_1\|_{1,\Omega_1} \lesssim |\tilde{u}_1\|_{1,\Omega 1} + \|\bar{u}_1\|_{1,\Omega_1} \lesssim \|\tilde{u}_1\|_{1,\Omega_1} + \|\bar{u}\|_{1/2,\Gamma}$$
$$\lesssim \frac{1}{\beta_1}\|f\|_{-1,\Omega_1} + \frac{1}{\beta_2}(\|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega}),$$

or

$$\beta_1\|u_1\|_{1,\Omega_1} \lesssim \|f\|_{-1,\Omega_1} + k(\|g\|_{-1/2,\Gamma} + \|f\|_{-1,\Omega}).$$

The desired result then follows immediately. ∎

Now, with a simple example below, we show that the jump k are indispensable in the estimates (2.36) and (2.38). Consider the following interface problem:

$$\begin{aligned} &-\Delta u_i = -1/\beta_i \qquad \text{in } \Omega_i, \quad i = 1, 2,\\ &[u] = 0, \qquad [\beta\partial_{\mathbf{n}} u] = 0 \qquad \text{on } \Gamma,\\ &u = 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega_1 = B_{1/2}(0)$ and $\Omega = B_1(0)$, with $B_r(0)$ being a ball of radius r centered at 0. In view of the symmetry of the domain and the conditions, we can assume $u(x_1, x_2, x_3)|_{\Omega_i} = u_i(r)$, $i = 1, 2$, under the polar coordinate system. Then the above interface problem reduces to

$$\begin{aligned} &\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = 1/\beta_1, \qquad 0<r<1/2,\\ &\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = 1/\beta_2, \qquad 1/2<r<1,\\ &[u]|_{r=1/2} = 0, \qquad [\beta u'(r)]|_{r=1/2} = 0,\\ &u(r)|_{r=1} = 0. \end{aligned}$$

By a straightforward computation we find

$$\begin{aligned} u_1 = u|_{\Omega_1} &= \frac{1}{6\beta_1}r^2 - \left(\frac{1}{24\beta_1} + \frac{3}{24\beta_2}\right),\\ u_2 = u|_{\Omega_2} &= \frac{1}{6\beta_2}r^2 - \frac{1}{6\beta_2}, \end{aligned}$$

which yields

$$\beta_1/\beta_2 \lesssim \beta_1\|u_1\|_{2,\Omega_1} \lesssim \beta_1/\beta_2, \qquad 1 \lesssim \beta_2\|u_2\|_{2,\Omega_2} \lesssim 1,$$

and

$$\beta_1/\beta_2 \lesssim \beta_1 \|u_1\|_{1,\,\Omega_1} \lesssim \beta_1/\beta_2, \qquad 1 \lesssim \beta_2 \|u_2\|_{1,\,\Omega_2} \lesssim 1.$$

This example verifies that in the case with $\beta_2 \ll \beta_1$, the coefficient β_2 has influence not only on the behavior of the solution in the domain Ω_2 but also strongly on the behavior of the solution in the domain Ω_1. This justifies our conclusions in Corollary 2.1 and Theorem 2.2.

3. UNIFORM A PRIORI ESTIMATES FOR PARABOLIC INTERFACE PROBLEMS

In this section, we consider the following parabolic interface problem:

$$\partial_t u - \nabla \cdot (\beta(x) \nabla u) = f(x, t) \qquad \text{in } Q_T = \Omega \times (0, T) \tag{3.40}$$

with the initial and boundary conditions

$$u(x, 0) = u_0(x) \qquad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \times (0, T) \tag{3.41}$$

and the jump conditions on the interface Γ

$$[u] = 0, \qquad [\beta(x) \partial_{\mathbf{n}} u] = g(x, t) \quad \text{across } \Gamma \times (0, T), \tag{3.42}$$

where $\beta(x)$ is positive and piecewise constant, that is,

$$\beta(x) = \beta_1, \qquad x \in \Omega_1, \qquad \beta(x) = \beta_2, \qquad x \in \Omega_2.$$

Here $\Omega,\ \Omega_1,\ \Omega_2$ and Γ are the same as stated in Section 1.

We first introduce some notations. For a Banach space B, we define

$$H^m(0, T; B) = \{u(t) \in B \text{ for a.e. } t \in (0, T) \text{ and } \|u\|_{H^m(0, T; B)} < \infty\},$$

where $\|u\|_{H^m(0, T; B)}$ is the norm of $H^m(0, T; B)$ and given by

$$\|u\|_{H^m(0, T; B)} = \left\{ \sum_{k=0}^{m} \int_0^T \|u^{(k)}(t)\|_B^2 \, dt \right\}^{1/2}.$$

Define $H^{2,1}(Q_T) = H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$, and its norm by

$$\|u\|_{H^{2,1}(Q_T), \lambda} = \{\lambda^2 \|u\|^2_{L^2(0, T; H^2(\Omega))} + \|u\|^2_{H^1(0, T; L^2(\Omega))}\}^{1/2},$$

where λ is a given parameter. We also define

$$X = H^2(\Omega_1) \cap H^2(\Omega_2) \cap H_0^1(\Omega),$$

$$Y = H^1(\Omega_1) \cap H^1(\Omega_2) \cap L^2(\Omega).$$

It was shown in [5, 16] that if $f \in H^1(0, T; L^2(\Omega))$, $g \in L^2(0, T; H^{1/2}(\Gamma))$ and $u_0 \in H^1(\Omega)$, then the parabolic interface problem (3.40)–(3.42) has a unique solution $u \in L^2(0, T; X) \cap H^1(0, T; Y)$. In this section, we attempt to present some uniform a priori estimates for the solution u of (3.40)–(3.42), similar to the a priori estimates obtained in Section 2.

THEOREM 3.1. *The solution of the interface problem* (3.40)–(3.42) *satisfies the a priori estimates*

$$\|u_1\|_{H^{2,1}(Q_T^1)), \beta_1} \lesssim \| f\|_{0, Q_T} + \|g\|_{L^2(0, T; H^{1/2}(\Gamma))} + \beta_i^{1/2}\|\nabla u_0\|_{0, \Omega_i}, \tag{3.43}$$

$$\|u_2\|_{H^{2,1}(Q_T^2), \beta_2} \lesssim \| f\|_{0, Q_T} + \|g\|_{L^2(0, T; H^{1/2}(\Gamma))} + \beta_i^{1/2}\|\nabla u_0\|_{0, \Omega_i} \tag{3.44}$$

if $\beta_1 < \beta_2$, *and*

$$\|u_1\|_{H^{2,1}(Q_T^1)),\beta_1} \lesssim k(\| f\|_{0, Q_T} + \|g\|_{L^2(0, T; H^{1/2}(\Gamma))} + \beta_i^{1/2}\|\nabla u_0\|_{0, \Omega_i}), \tag{3.45}$$

$$\|u_2\|_{H^{2,1}(Q_T^2)),\beta_2} \lesssim \| f\|_{0, Q_T} + \|g\|_{L^2(0, T; H^{1/2}(\Gamma))} + \beta_i^{1/2}\|\nabla u_0\|_{0, \Omega_i}, \tag{3.46}$$

if $\beta_2 < \beta_1$. *Here* $Q_T^1 = \Omega_1 \times (0, T)$, $Q_T^2 = \Omega_2 \times (0, T)$.

Proof. We only consider the case that $\beta_2 < \beta_1$, the other case can be handled similarly. It follows from (3.40)–(3.42) that, for a.e. $t \in (0, T)$, $u = u(x, t)$ solves

$$\begin{aligned} &-\nabla \cdot (\beta \nabla u) = f - \partial_t u \qquad \text{in } \Omega,\\ &u = 0 \qquad \text{on } \partial\Omega,\\ &[u] = 0, \qquad [\beta \partial_{\mathbf{n}} u] = g \qquad \text{on } \Gamma. \end{aligned} \tag{3.47}$$

Then, by Corollary 2.1 we know

$$\beta_1 \|u_1\|_{2, \Omega_1} \lesssim k(\|f - \partial_t u\|_{0, \Omega} + \|g\|_{1/2, \Gamma}), \tag{3.48}$$

$$\beta_2 \|u_2\|_{2, \Omega_2} \lesssim \|f - \partial_t u\|_{0, \Omega} + \|g\|_{1/2, \Gamma}. \tag{3.49}$$

It should be emphasized that the generic constants in (3.48) and (3.49) are independent of the time variable t. Taking the $L^2(\Omega)$-inner product with $\partial_t u$ on both sides of (3.40) gives

$$\|\partial_t u\|_{0,\,\Omega}^2 - (\nabla \cdot (\beta(x)\nabla u), \partial_t u) = \int_\Omega f \partial_t u\, dx. \tag{3.50}$$

Since $u \in H^1(0, T; Y)$ and $[u] = 0$ on Γ, we know

$$[\partial_t u] = 0 \qquad \text{on } \Gamma. \tag{3.51}$$

By the integration by parts and (3.51) we then obtain

$$\begin{aligned}
(\nabla \cdot (\beta(x)\nabla u), \partial_t u) &= \sum_{i=1}^{2} \beta_i(\nabla \cdot \nabla u_i, \partial_t u_i) \\
&= -\sum_{i=1}^{2} \beta_i(\nabla u_i, \partial_t \nabla u_i) - \int_\Gamma \partial_t u[\beta \partial_{\mathbf{n}} u]\, ds \\
&= -\frac{1}{2}\partial_t\left(\sum_{i=1}^{2} \beta_i \|\nabla u_i\|_{0,\,\Omega_i}^2\right) - \int_\Gamma \partial_t u g\, ds.
\end{aligned}$$

This together with (3.50) yields

$$\|\partial_t u\|_{0,\,\Omega}^2 + \frac{1}{2}\partial_t\left(\sum_{i=1}^{2} \beta_i \|\nabla u_i\|_{0,\,\Omega_i}^2\right) = \int_\Omega f \partial_t u\, dx - \int_\Gamma \partial_t u g\, ds,$$

and further by Cauchy–Schwarz inequality, we obtain

$$\|\partial_t u\|_{0,\,\Omega}^2 + \frac{1}{2}\partial_t\left(\sum_{i=1}^{2} \beta_i \|\nabla u_i\|_{0,\,\Omega_i}^2\right) \lesssim \|f\|_{0,\,\Omega}^2 + \|g\|_{0,\Gamma}^2. \tag{3.52}$$

Integrating both sides of (3.52) with respect to t from 0 to T implies

$$\begin{aligned}
&\|\partial_t u\|_{0,\,Q_T}^2 + \tfrac{1}{2}\{\beta_1\|\nabla u_1(\cdot, T)\|_{0,\,\Omega_1}^2 + \beta_2\|\nabla u_2(\cdot, T)\|_{0,\,\Omega_2}^2\} \\
&\quad \lesssim \|f\|_{0,\,Q_T}^2 + \|g\|_{L^2(0,T;L^2(\Gamma))}^2 \\
&\qquad + \tfrac{1}{2}(\nabla\beta_1\|u_1(\cdot,0)\|_{0,\,\Omega_1}^2 + \beta_2\|\nabla u_2(\cdot,0)\|_{0,\,\Omega_2}^2).
\end{aligned} \tag{3.53}$$

Now it follows from (3.48), (3.49) and (3.53) that

$$\begin{aligned}\beta_1 \|u_1\|_{L^2(0,T;H^2(\Omega_1))} &\lesssim k\{\|f\|_{0,Q_T} + \|\partial_t u\|_{0,Q_T} + \|g\|_{L^2(0,T;H^{1/2}(\Gamma))}\} \\ &\lesssim k\{\|\partial_t u\|^2_{0,\Omega} + \|g\|_{L^2(0,T;H^{1/2}(\Gamma))} \\ &\quad + \sum_{i=1}^{2} \beta_i^{1/2} \|\nabla u_i(\cdot,0)\|_{0,\Omega_i}\},\end{aligned} \tag{3.54}$$

$$\begin{aligned}\beta_2 \|u_2\|_{L^2(0,T;H^2(\Omega_2))} &\lesssim \|f\|_{0,Q_T} + \|\partial_t u\|_{0,Q_T} + \|g\|_{L^2(0,T;H^{1/2}(\Gamma))} \\ &\lesssim \|\partial_t u\|^2_{0,\Omega} + \|g\|_{L^2(0,T;H^{1/2}(\Gamma))} \\ &\quad + \sum_{i=1}^{2} \beta_i^{1/2} \|\nabla u_i(\cdot,0)\|_{0,\Omega_i}.\end{aligned} \tag{3.55}$$

Moreover, it follows from (3.40) that

$$\partial_t u_i = f + \beta_i \Delta u_i \qquad \text{in } \Omega_i, \quad i = 1, 2,$$

which by (3.54) and (3.55) implies

$$\|\partial_t u_j\|_{L^2(0,T;L^2(\Omega_j))} \lesssim \|f\|_{0,Q_T} + \sum_{i=1}^{2} \beta_i^{1/2} \|\nabla u_0\|_{0,\Omega_i}, \qquad j = 1, 2. \tag{3.56}$$

So we have proved (3.45) and (3.46). ∎

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